A Simple Way of Increasing Estimation Accuracy of Generalized Adaptive Notch Filters

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***Abstract*—Generalized adaptive notch filters are used for identification/tracking of quasi-periodically varying dynamic systems and can be considered an extension, to the system case, of classical adaptive notch filters. It is shown that frequency biases, which arise in generalized adaptive notch filtering algorithms, can be significantly reduced by incorporating in the adaptive loop an appropriately chosen decision delay. The resulting performance gains can be substantial. The proposed method can be used both in the system case and in the signal case.**

***Index Terms*—Adaptive notch filtering, frequency estimation, system identification, time-varying processes.**

I. Introduction

CONSIDER the problem of identification/tracking of a complex-valued quasi-periodically varying system governed by

$$y(t)=\sum_{l=1}^{n}\theta_l(t)\varphi_l(t)+v(t)=\boldsymbol{\varphi}^{\mathrm{T}}(t)\boldsymbol{\theta}(t)+v(t) \tag{1}$$

where $t=1,2,\ldots$ denotes the normalized discrete time, $y(t)$ denotes the system output, $\boldsymbol{\varphi}(t)=[\varphi_1(t),\ldots,\varphi_n(t)]^{\mathrm{T}}$ is the regression vector, $v(t)$ denotes complex-valued circular white measurement noise, and $\boldsymbol{\theta}(t)=[\theta_1(t),\ldots,\theta_n(t)]^{\mathrm{T}}$ denotes the vector of time-varying coefficients, modeled as weighted sums of complex exponentials

$$\theta_l(t)=\sum_{i=1}^{k}a_{li}(t)e^{j\sum_{s=1}^{t}\omega_i(s)}, \qquad l=1,\ldots,n. \tag{2}$$

We will assume that the complex amplitudes $a_{li}(t)$ and the instantaneous angular frequencies $\omega_i(t)\in[-\pi,\pi]$ in (2) may slowly vary with time. Under such conditions, the system described above changes in a periodic-like but not exactly periodic manner. Models such as (1) and (2) can be used, among others, for the purpose of equalization of rapidly fading telecommunication channels [1], [2].

Denote by $\boldsymbol{\alpha}_i(t)=[a_{1i}(t),\ldots,a_{ni}(t)]^{\mathrm{T}}$ the vector of system coefficients associated with a particular instantaneous frequency $\omega_i(\cdot)$ and let $\boldsymbol{\beta}_i(t)=f_i(t)\boldsymbol{\alpha}_i(t)$, where $f_i(t)=e^{j\sum_{s=1}^{t}\omega_i(s)}$. Observe that $\boldsymbol{\theta}(t)=\sum_{i=1}^{k}\boldsymbol{\beta}_i(t)$.

Identification algorithms capable of tracking the system in (1) and (2) were proposed in [3]–[5]. When the sequence of regression vectors $\{\boldsymbol{\varphi}(t)\}$ is wide-sense stationary and persistently exciting, with known covariance matrix $\boldsymbol{\Phi}=\mathrm{E}[\boldsymbol{\varphi}^*(t)\boldsymbol{\varphi}^{\mathrm{T}}(t)]>0$, the normalized steady-state version of the generalized adaptive notch filter (GANF) algorithm presented in [4] can be written down in the form

$$\begin{aligned}
\varepsilon(t)&=y(t)-\boldsymbol{\varphi}^{\mathrm{T}}(t)\sum_{i=1}^{k}e^{j\widehat{\omega}_i(t)}\widehat{\boldsymbol{\beta}}_i(t-1)\\
\widehat{\boldsymbol{\beta}}_i(t)&=e^{j\widehat{\omega}_i(t)}\widehat{\boldsymbol{\beta}}_i(t-1)+\mu\boldsymbol{\Phi}^{-1}\boldsymbol{\varphi}^*(t)\varepsilon(t)\\
g_i(t)&=\mathrm{Im}\left[\frac{\varepsilon^*(t)e^{j\widehat{\omega}_i(t)}\boldsymbol{\varphi}^{\mathrm{T}}(t)\widehat{\boldsymbol{\beta}}_i(t-1)}{\widehat{\boldsymbol{\beta}}_i^{\mathrm{H}}(t-1)\boldsymbol{\Phi}\widehat{\boldsymbol{\beta}}_i(t-1)}\right]\\
\widehat{\omega}_i(t+1)&=\widehat{\omega}_i(t)-\gamma g_i(t)\\
i&=1,\ldots,k\\
\widehat{\boldsymbol{\theta}}(t)&=\sum_{i=1}^{k}\widehat{\boldsymbol{\beta}}_i(t).
\end{aligned} \tag{3}$$

Tracking properties of this algorithm are determined by two user-dependent tuning coefficients: the adaptation gain $0<\mu\ll 1$, which controls the rate of amplitude adaptation, and another adaptation gain $0<\gamma\ll\mu$, which decides upon the rate of frequency adaptation.

The efficient initialization procedure, which can be used to identify the number of frequency modes and to determine initial conditions needed to smoothly start (i.e., start without initialization transients) GANF algorithms, was presented in [4].

When the instantaneous frequencies are constant, identification of (1) can be carried out using either the basis function (parametric) approach [2], [6] or the nonparametric cumulant-based methods, which exploit almost-cyclostationarity of the investigated system [7]. However, both approaches fail when the frequencies $\omega_1,\ldots,\omega_k$ change over time. Note that the analyzed system is "doubly nonstationary"—not only do its parameters vary in a sinusoidal (and possibly very fast) manner, but also, the rates of these sinusoidal changes are time-varying. To the best of our knowledge, GANF algorithms described in [3]–[5] are the first and the only ones that can successfully cope with tracking of quasi-periodically varying systems such as the one in (1) and (2)—see [3] for more comments on this issue.

In a special case where $n=1$ and $\varphi(t)=1,\forall t$, the model (1) in (2) becomes a description of a noisy nonstationary multifrequency signal, and generalized adaptive notch filters turn into "ordinary" adaptive notch filters (ANFs)—the algorithms

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used for extraction or elimination of sinusoidal signals buried in noise—see, e.g., [8], [9], and the references therein. ANFs have many applications such as cancellation of sinusoidal interferences or adaptive line enhancement.

II. Proposed Solution

Consider a quasi-periodically varying system with frequency modes governed by

$$\beta_i(t) = e^{j\omega_i(t)}\beta_i(t-1), \qquad i = 1,\dots,k. \tag{4}$$

Note that under (4), parameter changes of the analyzed system can be attributed exclusively to changes of the instantaneous frequencies (this is *not* a critical assumption—see [9] and the simulation example in Section III). Let $b_i^2 = \boldsymbol{\beta}_i^{\mathrm{H}}(t)\boldsymbol{\Phi}\boldsymbol{\beta}_i(t) = \boldsymbol{\beta}_i^{\mathrm{H}}(0)\boldsymbol{\Phi}\boldsymbol{\beta}_i(0)$ and $z_i(t) = \mathrm{Im}[\boldsymbol{\beta}_i^{\mathrm{H}}(t)\boldsymbol{\varphi}^*(t)v(t)/b_i^2]$. Since for a circular white noise $v(t) = v_R(t) + jv_I(t)$, it holds that $\mathrm{E}[v_R^2(t)] = \mathrm{E}[v_I^2(t)] = \sigma_v^2/2$, $\mathrm{E}[v_R(t)v_I(t)] = 0$, $\forall t$, one can easily show that $\{z_i(t)\}$ is a real-valued white noise with variance $\sigma_{z_i}^2 = \sigma_v^2/(2b_i^2)$.

If the sequence of regression vectors $\{\boldsymbol{\varphi}(t)\}$, independent of $\{v(t)\}$ and $\{\omega_i(t)\}$, is wide-sense stationary and persistently exciting, and if the estimated frequencies are well separated, namely, if $|\omega_i(t)-\omega_j(t)| > 2\mu, \forall\, i \neq j, \forall t$, then the relationship between $\widehat{\omega}_i(t)$ and $\omega_i(t)$ for the system governed by (4) can be approximately described by the following linear equation [5]:[1]

$$\widehat{\omega}_i(t) \cong F_1(q^{-1})z_i(t) + F_2(q^{-1})\omega_i(t) \tag{5}$$

$$F_1(q^{-1}) = \frac{(1-\delta)(1-q^{-1})q^{-1}}{1-(\lambda+\delta)q^{-1}+\lambda q^{-2}}$$

$$F_2(q^{-1}) = \frac{(1-\delta)q^{-1}}{1-(\lambda+\delta)q^{-1}+\lambda q^{-2}} \tag{6}$$

where q^{-1} denotes the backward shift operator and $\lambda = 1-\mu$, $\delta = 1-\gamma$.

Denote by $\bar{\omega}_i(t) = \mathrm{E}[\widehat{\omega}_i(t)|\omega_i(s), s \le t] = F_2(q^{-1})\omega_i(t)$ the mean path of frequency estimates for a particular frequency trajectory. Since $F_2(q^{-1})$ is a lowpass filter, for slowly varying instantaneous frequencies, the estimation bias $\bar{\omega}_i(t) - \omega_i(t)$ is dominated by lag errors: the mean trajectory of frequency estimates lags behind the true frequency trajectory. Roughly speaking, this means that $\widehat{\omega}_i(t)$ can be viewed as an estimate of $\omega_i(t-\tau_o)$ rather than of $\omega_i(t)$, where

$$\tau_o = -\lim_{\xi\mapsto 0}\frac{d\phi(\xi)}{d\xi} = \frac{\mu}{\gamma}$$

denotes a nominal (low-frequency) delay introduced by the filter $F_2(e^{-j\xi}) = A(\xi)e^{j\phi(\xi)}$ (to avoid confusion with $\omega_i(t)$, the standard Fourier-domain angular frequency variable was denoted by ξ). The nominal delay is usually a fairly good approximation of the optimal delay, defined as a time shift that minimizes the mean-squared value of the frequency matching error $\Delta\widehat{\omega}_i(t,\tau) = \widehat{\omega}_i(t) - \omega_i(t-\tau)$. Note that the optimal delay depends on the actual shape of the estimated frequency trajectory, which is not known in practice.

In order to increase estimation accuracy of GANF, one can run—as a follow up to (3)—another algorithm that incorporates the "debiased" (time shifted) frequency estimates

$$\begin{aligned}
\widetilde{\varepsilon}(t-l_o) &= y(t-l_o) - \boldsymbol{\varphi}^{\mathrm{T}}(t-l_o)\sum_{i=1}^{k} e^{j\widehat{\omega}_i(t)}\widetilde{\boldsymbol{\beta}}_i(t-l_o-1)\\
\widetilde{\boldsymbol{\beta}}_i(t-l_o) &= e^{j\widehat{\omega}_i(t)}\widetilde{\boldsymbol{\beta}}_i(t-l_o-1) + \mu\boldsymbol{\Phi}^{-1}\boldsymbol{\varphi}^*(t-l_o)\widetilde{\varepsilon}(t-l_o)\\
&\quad i = 1,\dots,k\\
\widetilde{\boldsymbol{\theta}}(t-l_o) &= \sum_{i=1}^{k}\widetilde{\boldsymbol{\beta}}_i(t-l_o)
\end{aligned} \tag{7}$$

where $l_o = \mathrm{int}[\tau_o] = \mathrm{int}[\mu/\gamma]$.

Note that the "frequency-guided" GANF filter (7) relies on frequency estimates provided by the "pilot" algorithm (3) and that it operates on a time-delayed input data sequence. The resulting decision delay of l_o sampling intervals is acceptable in the majority of signal processing applications, such as adaptive line enhancement or adaptive noise canceling, as well as in system tracking applications, such as equalization of telecommunication channels.

For a special class of polynomial phase systems, a more elaborate (but also less reliable) *delay-free* debiasing technique was recently proposed in [10].

The idea of reducing estimation bias by means of incorporating in the processing loop an appropriately chosen decision delay can be traced back to Hedelin [11]. Hedelin demonstrated that delaying state estimates provided by the Kalman filter can be regarded as a very efficient (often suboptimal) form of smoothing. The same technique can be used to improve tracking properties of adaptive filters—see [12] and [13] for a more detailed discussion of this approach.

III. Tracking Analysis

To gain insight into the tracking behavior of GANF, suppose that the sequence $\{\omega_i(t)\}$ is a zero-mean stationary stochastic process with spectral density function $S_{\omega_i}(\xi)$. Note that the frequency estimation error for the "pilot" algorithm (3) is equal to $\Delta\widehat{\omega}_i(t,0)$ and that the analogous error for the corrected, "frequency-guided" algorithm (7) can be expressed as $\Delta\widehat{\omega}_i(t,l_o)$. According to (5), it holds that

$$\Delta\widehat{\omega}_i(t,\tau) \cong F_1(q^{-1})z_i(t) + \left[F_2(q^{-1}) - q^{-\tau}\right]\omega_i(t)$$

and hence, using standard results from the linear filtering theory [14], one arrives at

$$\mathrm{E}\left[(\Delta\widehat{\omega}_i(t,\tau))^2\right] \cong \frac{1}{2\pi}\int_{-\pi}^{\pi} V(\xi)S_{z_i}(\xi)d\xi + \frac{1}{2\pi}\int_{-\pi}^{\pi} B(\xi,\tau)S_{\omega_i}(\xi)d\xi \tag{8}$$

where $S_{z_i}(\xi) = \sigma_{z_i}^2$, $\xi \in [-\pi,\pi]$,

$$V(\xi) = \left|F_1(e^{-j\xi})\right|^2, \quad B(\xi,\tau) = \left|F_2(e^{-j\xi}) - e^{-j\xi\tau}\right|^2$$

[1]The derivation presented in [5] is constrained to systems with a single frequency mode. Its extension to the multiple frequency case is possible by exploiting the frequency decoupling property of narrowband filters, which holds if the bandwidths of the component GANFs do not overlap [3].

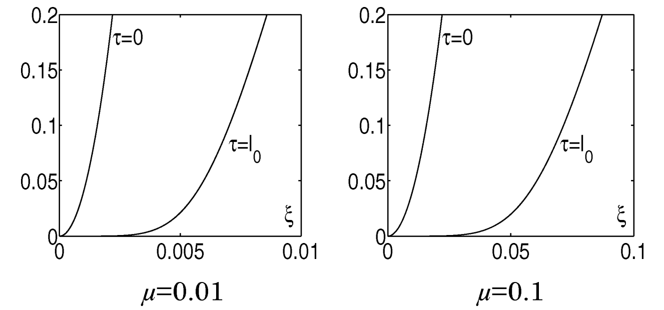

Fig. 1. Bias characteristics $B(\xi, \tau)$ for the original GANF algorithm ($\tau = 0$) and its bias-compensated version ($\tau = l_o$) for two different values of $\mu(\gamma = \mu^2/2)$. Note the horizontal scale difference between the left figure and the right figure.

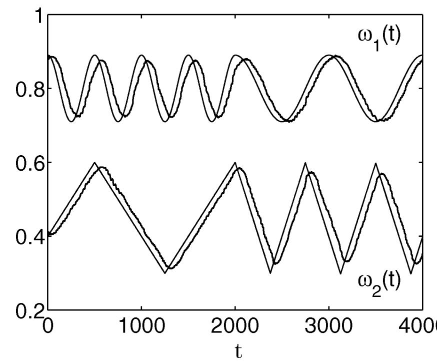

Fig. 2. True system frequency changes (thin lines) and typical trajectories of frequency estimates (thick lines) yielded by the GANF algorithm ($\mu = 0.03$, $\gamma = 0.00045$).

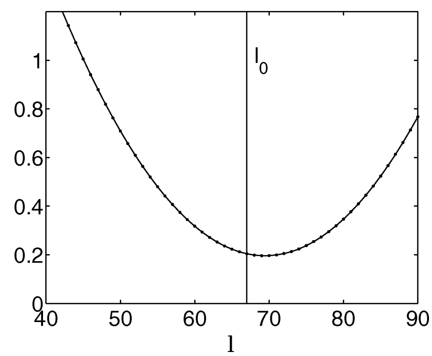

Fig. 3. Dependence of the accumulated mean-squared frequency matching error on the delay l; the nominal delay is marked with a vertical line.

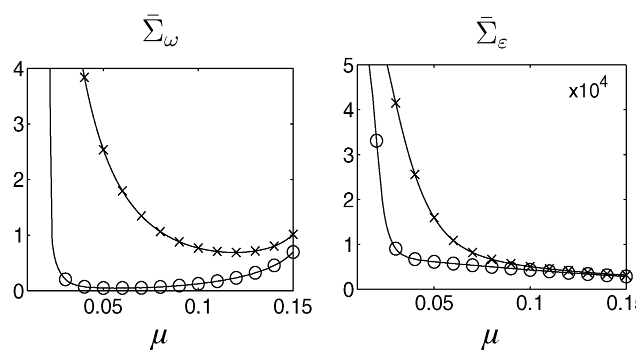

Fig. 4. Dependence of the averaged sums of the squared frequency estimation errors $\bar{\Sigma}_\omega$ and excess prediction errors $\bar{\Sigma}_\varepsilon$ on the adaptation gain μ. Comparison involves the estimates yielded by the original GANF algorithm ($\times$) and by its frequency-debiased version proposed in this letter ($\circ$). All plots were evaluated on a grid of 100 equidistant values of μ.

and the expectation is taken over different realizations of the measurement noise and different realizations of frequency trajectory.

The first term on the right-hand side of (8) can be interpreted as the variance component of the mean-squared tracking error and the second term as its bias component.

Note that the relationship (8), which specifies frequency distribution of the mean-squared matching error, clearly distinguishes between the user-dependent tracking characteristics of the GANF algorithm $(V(\xi), B(\xi, \tau))$ and characteristics of the identified nonstationary system $(S_{z_i}(\xi), S_{\omega_i}(\xi))$. It is obvious that good tracking performance can be achieved only in the case where the dominant part of the spectral density function $S_{\omega_i}(\xi)$ falls into the stopband region of $B(\xi, \tau)$, i.e., when the bandwidth of system frequency changes is matched by the tracking bandwidth of GANF. The shape of $B(\xi, \tau)$ is therefore a good guideline for comparing tracking capabilities of the algorithms described in Sections I and II.

Fig. 1 shows the plots of the bias characteristics $B(\xi, 0)$ and $B(\xi, l_o)$ for two values of μ: $\mu = 0.01$ and $\mu = 0.1$. To reduce the number of design degrees of freedom, the gain γ was set to $\mu^2/2$—see [5]. The plots clearly show advantages of the proposed approach—irrespective of the choice of μ, frequency debiasing allows one to significantly widen (approximately four times) the stopband of the bias characteristic and hence to reduce bias errors. Since the variance characteristics $V(\xi)$ are in both cases the same, the two-step algorithm (3) plus (7) will be always more accurate than the basic algorithm (3).

Figs. 2–4 summarize results of a simulation experiment arranged to check tracking capabilities of the proposed algorithm. The simulated system, inspired by channel estimation applications, was governed by $y(t) = \theta_1(t)u(t) + \theta_2(t)u(t-1) + v(t)$, where $\theta_l(t) = \sum_{i=1}^{2} a_{li}(t)e^{j\sum_{s=1}^{t}\omega_i(s)}$, $l = 1, 2$, i.e., it was a two-tap FIR system ($n = 2$) with two modes of parameter variation ($k = 2$). The weighting coefficients were time-varying: $a_{11}(t) = (2 - j)g_1(t)$, $a_{12}(t) = (1 + 2j)g_2(t)$, $a_{21}(t) = (1 - 2j)g_2(t)$, $a_{22}(t) = (2 + j)g_1(t)$, where $g_1(t) = 1 + 0.5\sin(2\pi t/1000)$ and $g_2(t) = 1 + 0.5\cos(2\pi t/1000)$. The white 4-QAM sequence was used as the input signal $(u(t) = \pm 1 \pm j, \sigma_u^2 = 2)$, and the noise was circular complex Gaussian with variance $\sigma_v^2 = 1$.

Fig. 2 shows the evolution of the instantaneous frequencies $\omega_1(t)$ and $\omega_2(t)$ and trajectories of the corresponding frequency estimates obtained from the regular GANF algorithm (3) for $\mu = 0.03$ and $\gamma = 0.00045$. Note a clearly visible delay between the true trajectories and the estimated trajectories. Fig. 3 shows how the accumulated mean-squared frequency matching error, evaluated over the interval [1001, 3000], depends on the choice of the delay $l = \mathrm{int}[\tau]$. In agreement with theory, the nominal delay equal to $l_o = \mathrm{int}[\mu/\gamma] = 67$ sampling intervals is close to the optimal delay (69 sampling intervals).

Fig. 4 allows one to compare the tracking performance of the regular algorithm and of its debiased version. Tracking capabilities of the compared algorithms were measured in terms of the accumulated frequency estimation errors

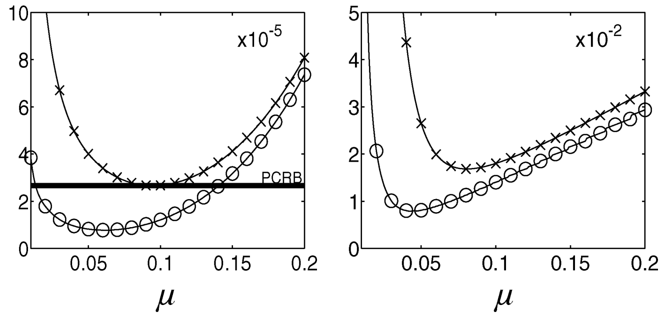

Fig. 5. Dependence of the mean-squared frequency matching errors (left figure) and excess prediction errors (right figure) on the adaptation gain $\mu(\gamma = \mu^2/2)$ for a system with a single frequency mode subject to random walk frequency variation. Comparison involves the estimates yielded by the original GANF algorithm ($\times$) and by its frequency-debiased version proposed in this letter ($\circ$). All plots were evaluated on a grid of 20 equidistant values of μ. The posterior Cramér–Rao bound is indicated by a horizontal line on the left figure.

$\Sigma_\omega = \sum_{t=1001}^{3000}\{[\widehat{\omega}_1(t)-\omega_1(t)]^2+[\widehat{\omega}_2(t)-\omega_2(t)]^2\}$ and the accumulated excess prediction errors $\Sigma_\varepsilon = \sum_{t=1001}^{3000}[|\varepsilon(t)|^2-\sigma_v^2]$, after the filters have reached their steady-state behavior. The plots show how ensemble averages of both error statistics (obtained for 25 different realizations of measurement noise) depend on the choice of μ (γ was set to $\mu^2/2$). As expected, delay compensation led to improved tracking results, both in terms of the minimum achievable errors and, more importantly, in terms of the algorithm's robustness to the choice of μ.

IV. Beyond the Posterior Cramér–Rao Bound

When the system in (1) and (2) has a single frequency mode ($k = 1$) governed by the random walk model, i.e., when the one-step frequency changes $w(t) = \omega(t) - \omega(t-1)$ form a white noise sequence of variance σ_w^2, the mean-squared frequency tracking error can be evaluated and minimized analytically. As shown in [5], in the Gaussian case, the minimum tracking error, obtained for the optimal settings $\mu_{\mathrm{opt}} = \sqrt[4]{8\kappa}$, $\gamma_{\mathrm{opt}} = \sqrt{2\kappa} = \mu_{\mathrm{opt}}^2/2$, $\kappa = b^2\sigma_w^2/\sigma_v^2$, attains its theoretical lower limit known as the posterior Cramér–Rao lower bound [15]

$$\mathrm{PCRB} = \inf_{\widehat{\omega}(\cdot)} \mathrm{E}\left[(\widehat{\omega}(t)-\omega(t))^2\right] \cong \sqrt[4]{2\kappa^{-1}}\sigma_w^2. \qquad (9)$$

This means that in the case considered, an optimally tuned GANF algorithm is a statistically efficient procedure for tracking of a randomly drifting frequency.

Fig. 5 shows dependence of the mean-squared frequency matching errors and excess prediction errors on μ for a single-tap FIR system governed by $y(t) = \theta(t)u(t) + v(t)$, $\theta(t) = ae^{j\sum_{s=1}^{t}\omega(s)}$, subject to random walk frequency variation ($a = (\sqrt{2}+j\sqrt{2})/2$, $\omega(0) = 0.4$, $\sigma_w^2 = 10^{-6}$, $\sigma_v^2 = 0.5$, $u(t) = \pm 1 \pm j$). All results were averaged over 25 realizations of $\{v(t), w(t)\}$. Note that the debiased GANF algorithm performs better than the regular GANF filter. Note also that the well-tuned debiased algorithm provides "sub-PCRB" performance. This is not a surprise. The posterior Cramér–Rao bound applies to causal estimation schemes, such as (3). When estimation is based on both "past" and "future" measurements, i.e., when some form of smoothing is incorporated (note that at each time instant, the debiased GANF algorithm makes use of l_o "future" samples), the "causal" Cramér–Rao bound (9), regarded as a theoretical limit to the achievable tracking performance, should be replaced with a smaller, "noncausal" bound (not established yet), adequate for smoothing.

V. Conclusion

We have shown that frequency biases, which arise in GANF algorithms, can be significantly reduced by incorporating in the adaptive loop a judiciously chosen decision delay. Such a delay is acceptable in many practical applications. The proposed solution is a cascade of two filters. The "pilot" GANF provides preliminary (biased) frequency estimates. The estimates yielded by the pilot algorithm are fed into the second algorithm—the "frequency-guided" GANF, which operates on a delayed data sequence. We have shown that frequency debiasing improves the tracking performance of GANF algorithms and increases their robustness to the choice of design parameters.

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